

Hermitian dynamics in a class of pseudo-Hermitian networks

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We investigate a pseudo-Hermitian lattice system, which consists of a set of isomorphic pseudo-Hermitian clusters coupled together in a Hermitian manner. We show that such non-Hermitian systems can act as Hermitian systems. This is made possible by considering the dynamics of a state involving an identical eigenmode of each isomorphic cluster. It still holds when multiple eigenmodes are involved when additional restriction on the state is imposed. This Hermitian dynamics is demonstrated for the case of an exactly solvable \mathcal{PT} -symmetric ladder system.

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I. INTRODUCTION

The Hermitian quantum mechanics is a well-developed framework because a Hermitian Hamiltonian leads to a real spectrum and unitary time evolution, which preserves the probability normalization. However, a decade ago it was observed that a large class of non-Hermitian Hamiltonians possess real spectra [1, 2] and a pseudo-Hermitian Hamiltonian connects with its equivalent Hermitian Hamiltonian via a similarity transformation [3, 4], quantum theory based on non-Hermitian Hamiltonian was established [4–13]. In additional, such a framework also indicates the preservation of probability normalization if a positive-definite inner product is employed to replace the Dirac inner product. Nevertheless, to date the interpretation and the measurement in experiment of such an inner product are not clear. While the Dirac probability (Dirac inner product) can be measured in a universal manner. Therefore being an acceptable theory of quantum mechanics, the Dirac probability is of central importance to most practical physical problems.

Parity and time-reversal symmetric (\mathcal{PT} -symmetric) system has attracted much attention due to recent progresses on experimental investigations in \mathcal{PT} -symmetric optical systems, observation of passive \mathcal{PT} -symmetric breaking in passive optical double-well structure [14] and observation of spontaneous symmetry breaking together with power oscillations in optical coupled system [15] were carried out. Optics offers the rather unique advantage in detection the wave function evolution and seem to be the most readily applicable [16, 17]. In the past two decades, general issues of quantum effects in quantum systems have proven to be successfully investigated in the framework of quantum optical analogy based on the fact that paraxial propagation of light in optical guided structures is governed by a Schrödinger-like equation [16]. Actually, the intensity observed in optical experiment corresponds to the Dirac probability of electric field envelope. It is not guaranteed for generic systems that the Dirac probability is preserved even when the non-Hermitian Hamiltonian is time inde-

pendent. Nevertheless, the violation of the conservation of Dirac probability in non-Hermitian system does not contradict the Copenhagen interpretation. The implications of pseudo-Hermitian system are still under consideration, peculiar features were exhibited such as double refraction, power oscillations, *etc.* [18, 19] following by the suggestion of realization of \mathcal{PT} -symmetric structure in the realm of optics [20], while nonreciprocal Bloch oscillation with no classical correspondence was also shown in \mathcal{PT} -symmetric complex crystal [21].

We propose a class of non-Hermitian lattice systems in this work, the system is composed by a set of isomorphic pseudo-Hermitian clusters, which connected with each other in a Hermitian way. We show that in such non-Hermitian systems, Hermitian like dynamics can be observed, including the property that the time evolution is Dirac probability preserving. This is made possible by considering the dynamics of a state involving the superposition of an identical eigenmode of each isomorphic cluster in general case. In the case of having additional orthogonal modes, it still holds when multiple eigenmodes are involved. This Hermitian dynamics, as well as the quasi-Hermitian behavior, are specifically demonstrated for the case of an exactly solvable pseudo-Hermitian system.

This paper is organized as follows. In Section II, we present the model we focus on and its basic properties. Section III consists of an exactly solvable example to illustrate our main idea. Section IV is the summary and discussion.

II. HAMILTONIAN AND BASIC PROPERTIES

A general tight-binding network is constructed topologically by the sites and the various connections between them. As a simplified model, it captures the essential features of many discrete systems. Also it is a nice testing ground for the study of the non-Hermitian quantum mechanics due to its analytical and numerical tractability. Much effort has been devoted in recent years to the pseudo-Hermitian lattice system [22–34]. The Hamiltonian of the concerned tight-binding network reads as follows

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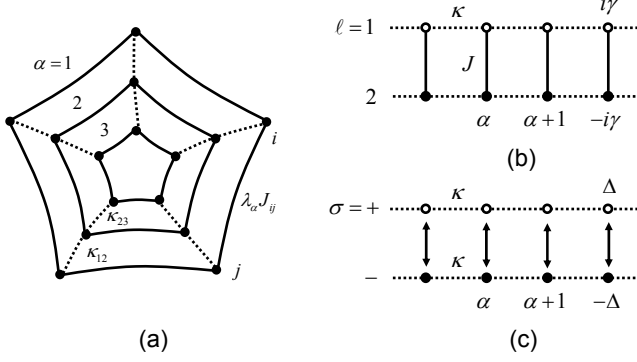


FIG. 1. Schematic illustration of the concerned networks. (a) A lattice consists of three 5-site isomorphic clusters, where the different sizes indicate the factor λ_α . The dot lines denote the similarity-mapping-type Hermitian structure couplings across the clusters. (b) A concrete example which is a two-leg ladder. Each rung is a non-Hermitian cluster. (c) Equivalent two-band model Eq. (24). Here the double-headed arrow denotes the quasi-canonical commutation relations between the eigenmodes $\sigma = \pm$ for the same cluster.

$$H = \sum_{\alpha=1}^N H_\alpha + \sum_{\alpha < \beta} H_{\alpha\beta}, \quad (1)$$

$$H_\alpha = \lambda_\alpha \sum_{i,j=1}^{N_d} J_{ij} a_{\alpha,i}^\dagger a_{\alpha,j}, \quad (2)$$

$$H_{\alpha\beta} = \kappa_{\alpha\beta} \sum_{l=1}^{N_d} a_{\alpha,l}^\dagger a_{\beta,l} + \text{H.c.}, \quad (3)$$

which consists of N isomorphic clusters H_α , with each cluster has a dimension N_d . The label α denotes the α th subgraph of N clusters, and $a_{\alpha,i}^\dagger$ ($a_{\alpha,i}$) is the boson or fermion creation (annihilation) operator at the i th site in the α th cluster. The cluster H_α is defined by the distribution of the hopping integrals $\{\lambda_\alpha J_{ij}\}$ where λ_α is real. The set of clusters are isomorphic due the fact that they have the same eigenfunctions and spectral structures. Note that terms $\sum_{\alpha < \beta} H_{\alpha\beta}$ is self-adjoint since $H_{\alpha\beta} = H_{\alpha\beta}^\dagger$, which describes the Hermitian connection between clusters. And such kind of couplings are the type of *similarity mapping*, which is crucial for the conclusion of this paper. The total Hamiltonian H is not Hermitian when the matrix J_{ij} is not Hermitian. Figure 1(a) shows a schematic example.

In this paper, we consider the case of H_α being pseudo-Hermitian, i.e., H_α is non-Hermitian but has entirely real spectrum. Then H is also pseudo-Hermitian in the case of real λ_α , hence possessing the common exceptional point as H_α . In general, a pseudo-Hermitian Hamiltonian does not guarantee the Dirac probability preserving. It has been shown that the Dirac norm of an evolved wavepacket ceases preserving as long as it touches the

region of on-site imaginary potentials [35]. In the following we will show due to the pseudo-Hermitian clusters combined together in a Hermitian way, that there exist quantum states obeying Dirac probability preserving, even if their profiles cover the imaginary potentials.

We start with the eigen problem of the Hamiltonian H_α . In single-particle invariant subspace, following the well established pseudo-Hermitian quantum mechanics [11–13], we always have

$$H_\alpha \bar{a}_{\alpha,\sigma} |\text{vac}\rangle = \lambda_\alpha \epsilon_\sigma \bar{a}_{\alpha,\sigma} |\text{vac}\rangle, \quad (4)$$

and

$$H_\alpha^\dagger a_{\alpha,\sigma}^\dagger |\text{vac}\rangle = \lambda_\alpha \epsilon_\sigma a_{\alpha,\sigma}^\dagger |\text{vac}\rangle, \quad (5)$$

where $\alpha \in [1, N]$ and $\sigma \in [1, N_d]$, the operators $\bar{a}_{\alpha,\sigma}$ and $a_{\alpha,\sigma}$ have the form

$$\bar{a}_{\alpha,\sigma} = \sum_l f_{l\sigma} a_{\alpha,l}^\dagger, \quad a_{\alpha,\sigma} = \sum_l g_{l\sigma}^* a_{\alpha,l}, \quad (6)$$

where

$$\sum_\sigma g_{l\sigma}^* f_{l'\sigma} = \delta_{ll'}, \quad \sum_l g_{l\sigma}^* f_{l\sigma'} = \delta_{\sigma\sigma'}. \quad (7)$$

Note that $\{f_{l\sigma}\}$, $\{g_{l\sigma}\}$ and $\{\epsilon_\sigma\}$ are independent of α . Then the operators $\bar{a}_{\alpha,\sigma}$ and $a_{\alpha,\sigma}$ are canonical conjugate pairs, satisfying

$$[a_{\alpha,\sigma}, \bar{a}_{\alpha',\sigma'}]_\pm = \delta_{\alpha\alpha'} \delta_{\sigma\sigma'}, \quad (8)$$

$$[a_{\alpha,\sigma}, a_{\alpha',\sigma'}]_\pm = [\bar{a}_{\alpha,\sigma}, \bar{a}_{\alpha',\sigma'}]_\pm = 0, \quad (9)$$

where $[\cdot, \cdot]_\pm$ denotes the the commutator and anti-commutator. And accordingly, the original Hamiltonian can be rewritten as the form

$$H = \sum_{\alpha,\sigma} \lambda_\alpha \epsilon_\sigma \bar{a}_{\alpha,\sigma} a_{\alpha,\sigma} + \sum_{\alpha < \beta, \sigma} (\kappa_{\alpha\beta} \bar{a}_{\alpha,\sigma} a_{\beta,\sigma} + \kappa_{\alpha\beta}^* \bar{a}_{\beta,\sigma} a_{\alpha,\sigma}), \quad (10)$$

which has the following subtle features: (i) The matrix representation of H with respect to the biorthogonal basis $\{|\text{vac}\rangle a_{\alpha,\sigma}, \bar{a}_{\alpha,\sigma} |\text{vac}\rangle\}$ is Hermitian, i.e., $\langle \text{vac} | a_{\alpha,\sigma} H \bar{a}_{\alpha',\sigma'} | \text{vac} \rangle = (\langle \text{vac} | a_{\alpha',\sigma'} H \bar{a}_{\alpha,\sigma} | \text{vac} \rangle)^*$; (ii) Although it is a non-Hermitian operator, i.e., $H \neq H^\dagger$, straightforward algebra shows that

$$[a_{\alpha,\sigma}, a_{\alpha',\sigma'}^\dagger]_\pm \propto \delta_{\alpha\alpha'}, \quad [a_{\alpha,\sigma}, a_{\alpha',\sigma'}]_\pm = 0, \quad (11)$$

which indicates that although with nonorthogonality of the eigenstates as inherent feature of non-Hermitian system, $a_{\alpha,\sigma}$ and $a_{\alpha',\sigma'}^\dagger$ obey quasi-canonical commutation relations due to the Hermitian connection structure between clusters. This results in a new type of particle

statistics, that is rarely observed in Hermitian systems, thus becomes highly relevant in the presence of non-Hermitian terms.

Considering an arbitrary state in the form

$$|\Phi_\sigma(0)\rangle = \sum_\alpha c_\alpha \bar{a}_{\alpha,\sigma} |\text{vac}\rangle, \quad (12)$$

as the initial state, where $\sum_\alpha |c_\alpha|^2 = 1$ and in which only the eigenmode σ of each cluster is involved. At instant t , we have

$$|\Phi_\sigma(t)\rangle = \sum_\alpha c_\alpha e^{-iHt} \bar{a}_{\alpha,\sigma} |\text{vac}\rangle. \quad (13)$$

In the framework of metric operator theory, H acts as a Hermitian system, obeying unitary time evolution in the positive-definite inner product [4]. However to date the physical meaning of the positive-definite inner product is unclear, while the Dirac probability can be measured in a universal manner, e.g. Dirac probability of wave electric field corresponds to the light intensity in optics and is simple to detect in experiment [16], therefore Dirac norm is of central importance. The aim of this paper is to show that contrary to the nonclassical dynamical behavior [18, 21], the unitary Dirac probability dynamics can also be observed in the pseudo-Hermitian system. Actually inserting $\sum_{\beta,\sigma'} \bar{a}_{\beta,\sigma'} |\text{vac}\rangle \langle \text{vac}| a_{\beta,\sigma'} = 1$ into Eq. (13), we have

$$\begin{aligned} |\Phi_\sigma(t)\rangle &= \sum_{\alpha,\beta} c_\alpha \bar{a}_{\beta,\sigma} |\text{vac}\rangle \langle \text{vac}| a_{\beta,\sigma} e^{-iHt} \bar{a}_{\alpha,\sigma} |\text{vac}\rangle \\ &= \sum_{\alpha,\beta} c_\alpha U_{\beta\alpha} \bar{a}_{\beta,\sigma} |\text{vac}\rangle, \end{aligned} \quad (14)$$

where

$$U_{\beta\alpha} = \langle \text{vac}| a_{\beta,\sigma} e^{-iHt} \bar{a}_{\alpha,\sigma} |\text{vac}\rangle, \quad (15)$$

is the propagator in the framework of biorthogonal basis and satisfies

$$\sum_\gamma U_{\gamma\alpha} U_{\gamma\beta}^* = \delta_{\alpha\beta}, \quad (16)$$

due to the above mentioned feature (i) of H . Accordingly, the Dirac norm has the form

$$\begin{aligned} \|\Phi_\sigma(t)\|^2 &= (|\Phi_\sigma(t)\rangle)^\dagger |\Phi_\sigma(t)\rangle \\ &= \left(\sum_{\alpha',\beta'} c_{\alpha'}^* U_{\beta'\alpha'}^* \langle \text{vac}| \bar{a}_{\beta',\sigma}^\dagger \right) \left(\sum_{\alpha,\beta} c_\alpha U_{\beta\alpha} \bar{a}_{\beta,\sigma} |\text{vac}\rangle \right) \\ &= \sum_\alpha |c_\alpha|^2 \Delta_\sigma = \Delta_\sigma, \end{aligned} \quad (17)$$

where the relation Eq. (16) is applied and the α -independent factor Δ_σ can be obtained from

$$\langle \text{vac}| \bar{a}_{\alpha,\sigma}^\dagger \bar{a}_{\beta,\sigma} |\text{vac}\rangle = \Delta_\sigma \delta_{\alpha\beta}. \quad (18)$$

It follows that although $|\Phi_\sigma(t)\rangle$ is not the eigenstate of the entire network system, the time evolution is Dirac norm-conserving, this is a direct consequence of the quasi-canonical commutation relations. The result presented here for the evolution of an arbitrary state involving an identical isomorphic-cluster-eigenmode provides a new way for connecting the pseudo-Hermitian and Hermitian systems.

It is worth to mention that this probability preserving evolution can also occur for a state involving multiple eigenmodes. This due to the fact that there always exist states, which parts belong to different eigenmodes are orthogonal in terms of Dirac inner product, hence preserve the Dirac probability. For instance, a state involves two eigenmodes σ_1 and σ_2 , its parts on σ_1 and σ_2 are spatially separated local states with respect to the coordinate space α , then the two parts of the state are orthogonal in terms of Dirac inner product and the evolution of such a state is probability preserving since the quasi-canonical commutation relations. We will demonstrate this point explicitly via the following illustrative example.

III. PSEUDO-HERMITIAN LADDER

Now we investigate a concrete example to demonstrate the application of the previous result. We consider a system of a two-leg ladder [Fig. 1(b)], consisting of N dimers as pseudo-Hermitian clusters. The Hamiltonian reads

$$H_{\text{Ladd}} = \sum_{\alpha=1}^N H_\alpha + \sum_{\alpha=1}^N H_{\alpha,\alpha+1}, \quad (19)$$

$$H_\alpha = -J(a_{\alpha,1}^\dagger a_{\alpha,2} + \text{H.c.}) + i\gamma(n_{\alpha,1} - n_{\alpha,2}), \quad (20)$$

$$H_{\alpha,\alpha+1} = -\kappa \sum_{\ell=1}^2 (a_{\alpha,\ell}^\dagger a_{\alpha+1,\ell} + \text{H.c.}), \quad (21)$$

where $n_{\alpha,\ell} = a_{\alpha,\ell}^\dagger a_{\alpha,\ell}$ is the particle number operator and the operators obey the periodic boundary condition $a_{N+1,\ell}^\dagger = a_{1,\ell}^\dagger$, with $\ell = 1, 2$. $\kappa(J)$ is the hopping integral along legs (rungs) and γ denotes the norm of the imaginary on-site potential. Note that the ladder is a \mathcal{PT} -symmetric Hamiltonian, where \mathcal{P} is the parity and \mathcal{T} denotes time-reversal. The simple structure of this model makes it an ideal testing ground for a more profound understanding of the Hermitian dynamics in a pseudo-Hermitian system. Taking the transformations

$$\bar{a}_{\alpha,\sigma} = \frac{1}{\sqrt{2\cos\theta}} \left(e^{i\sigma\theta/2} a_{\alpha,1}^\dagger - \sigma e^{-i\sigma\theta/2} a_{\alpha,2}^\dagger \right), \quad (22)$$

$$a_{\alpha,\sigma} = \frac{1}{\sqrt{2\cos\theta}} \left(e^{i\sigma\theta/2} a_{\alpha,1} - \sigma e^{-i\sigma\theta/2} a_{\alpha,2} \right), \quad (23)$$

where $(\alpha \in [1, N], \sigma = \pm)$, which are obtained from the solution of the dimer (a general solution of N_d -dimension

cluster is shown in Ref. [36].), the ladder Hamiltonian can be written as

$$H_{\text{Ladd}} = \sum_{\alpha=1, \sigma=\pm}^N (-\kappa \bar{a}_{\alpha, \sigma} a_{\alpha+1, \sigma} - \kappa \bar{a}_{\alpha+1, \sigma} a_{\alpha, \sigma} + \sigma \Delta \bar{a}_{\alpha, \sigma} a_{\alpha, \sigma}). \quad (24)$$

which is illustrated in Fig. 1(c), here $\Delta = \sqrt{J^2 - \gamma^2}$ and $\sin \theta = \gamma/J$, $\theta \in [0, \pi/2]$. The biorthogonal structure of the solution for a dimer admits the following canonical commutation relations Eq. (8) and

$$[\bar{a}_{\alpha, \sigma}^\dagger, \bar{a}_{\alpha', \sigma}]_\pm = [a_{\alpha, \sigma}, a_{\alpha', \sigma}^\dagger]_\pm = \sec \theta \delta_{\alpha\alpha'}, \quad (25)$$

$$[\bar{a}_{\alpha, -\sigma}^\dagger, \bar{a}_{\alpha', \sigma}]_\pm = [a_{\alpha, \sigma}, a_{\alpha', -\sigma}^\dagger]_\pm = i\sigma \tan \theta \delta_{\alpha\alpha'}. \quad (26)$$

Obviously, Hamiltonian Eq. (24) represents a two-band model, which has an interesting feature comparing to a Hermitian two-band model: although there are no inter-band transitions, the two bands are not independent. It is due to the pseudo-Hermiticity of the clusters, which allows $[a_{\alpha, \sigma}, \bar{a}_{\alpha, -\sigma}]_\pm = 0$ but $[a_{\alpha, \sigma}, a_{\alpha, -\sigma}^\dagger]_\pm \neq 0$. This characteristic will be further demonstrated through the following quasi-canonical commutation relations Eq. (33) and the time evolution for various Gaussian wavepackets. Figure 1(c) schematically illustrates such an equivalent two-band structure. Nevertheless, Hamiltonian Eq. (24) can be diagonalized as a Hermitian one, i.e., we have

$$H_{\text{Ladd}} = \sum_{k, \sigma=\pm} \varepsilon_{k, \sigma} \bar{a}_{k, \sigma} a_{k, \sigma}, \quad (27)$$

$$\varepsilon_{k, \pm} = -2\kappa \cos k \pm \Delta, \quad (28)$$

by using the linear transformations

$$\bar{a}_{k, \sigma} = \frac{1}{\sqrt{N}} \sum_{j=1}^N e^{ikj} \bar{a}_{j, \sigma}, \quad (29)$$

$$a_{k, \sigma} = \frac{1}{\sqrt{N}} \sum_{j=1}^N e^{-ikj} a_{j, \sigma}, \quad (30)$$

where $k = 2n\pi/N$, $n \in [1, N]$. The linearity of the transformations allows

$$[a_{k, \sigma}, \bar{a}_{k', \sigma'}]_\pm = \delta_{kk'} \delta_{\sigma\sigma'}, \quad (31)$$

$$[\bar{a}_{k, \sigma}, \bar{a}_{k', \sigma'}]_\pm = [a_{k, \sigma}, a_{k', \sigma'}]_\pm = 0. \quad (32)$$

However, when dealing with the Dirac inner product, the quasi-canonical commutation relations

$$[\bar{a}_{k, \sigma}^\dagger, \bar{a}_{k', \sigma}]_\pm = [a_{k, \sigma}, a_{k', \sigma}^\dagger]_\pm = \sec \theta \delta_{kk'}, \quad (33a)$$

$$[\bar{a}_{k, -\sigma}^\dagger, \bar{a}_{k', \sigma}]_\pm = [a_{k, \sigma}, a_{k', -\sigma}^\dagger]_\pm = i\sigma \tan \theta \delta_{kk'}, \quad (33b)$$

will be taken into account. Such quasi-canonical commutation relations reflect the subtle features of the system: when dealing with different k , $a_{k, \sigma}$ and $a_{k', \sigma'}^\dagger$ act as canonical conjugate pairs and the system displays Hermitian behavior.

We can gain some insight regarding the role of the quasi-canonical statistics. We will see shortly that such a model displays the similar dynamics as a Hermitian ladder. We start our investigation from the quantum dynamics of various initial wavepackets. In the situation of a Hermitian ladder, any two wavepackets propagate independently and the total probability is preserving.

Considering an arbitrary state involving both upper and lower bands

$$|\Phi(0)\rangle = \sum_{k, \sigma=\pm} f_{k, \sigma} \bar{a}_{k, \sigma} |\text{vac}\rangle, \quad (34)$$

with $\sum_{k, \sigma=\pm} |f_{k, \sigma}|^2 = 1$, we have

$$\begin{aligned} \|\Phi(t)\|^2 &= \sum_{k, \sigma} |f_{k, \sigma}|^2 \langle \text{vac} | [\bar{a}_{k, \sigma}^\dagger, \bar{a}_{k, \sigma}]_\pm | \text{vac} \rangle \\ &+ \sum_{k, \sigma} f_{k, -\sigma}^* f_{k, \sigma} e^{-i2\sigma\Delta t} \langle \text{vac} | [\bar{a}_{k, -\sigma}^\dagger, \bar{a}_{k, \sigma}]_\pm | \text{vac} \rangle \\ &= \sec \theta + i \tan \theta \sum_{k, \sigma} \sigma f_{k, -\sigma}^* f_{k, \sigma} e^{-i\sigma 2\pi(t/T_D)}, \end{aligned} \quad (35)$$

where $T_D = \pi/\Delta$ denotes the period of the oscillation. The first term gives the contribution from single band, while the second term captures the influence of the non-Hermiticity. For vanishing θ we recover the unitary evolution in Hermitian system. Evidently, $\|\Phi(t)\|^2 = \sec \theta$ for a state with $f_{k, -\sigma}^* f_{k, \sigma} = 0$, which involves only a single mode. Note, however, that mathematically speaking the time dependent terms can vanish even in the case of $f_{k, -\sigma}^* f_{k, \sigma} \neq 0$, e.g. additional orthogonality of the wavepacket with multiple eigenmodes. To demonstrate this, we study the evolution of initial wavepackets of the form

$$\begin{aligned} |\Psi(N_A, N_B, \phi_A, \phi_B, 0)\rangle &= \frac{1}{\sqrt{\Omega}} \\ &\times \sum_k \left[e^{-(k-\phi_A)^2/(2\rho^2)} e^{-i(k-\phi_A)N_A} \bar{a}_{k, +} \right. \\ &\left. + e^{-(k-\phi_B)^2/(2\rho^2)} e^{-i(k-\phi_B)N_B} \bar{a}_{k, -} \right] |\text{vac}\rangle, \end{aligned} \quad (36)$$

which is the superposition of wavepackets A and B , where $\Omega = 2 \sum_k e^{-(k-\phi_A)^2/\rho^2} = 2 \sum_k e^{-(k-\phi_B)^2/\rho^2}$. The time evolution of wavepacket is a powerful tool for understanding the dynamical property of Hermitian quantum systems [37]. Recently, The propagation of wavepacket in discrete systems has been utilized as flying qubit for quantum state transfer [38–42]. In the Hermitian case an initially Gaussian state stays Gaussian as it propagates for a long time, especially for the case of $|\phi_{A,B}| = \pi/2$ [43].

For a sufficient broad wavepacket ($\rho \ll 1$), we have $\Omega \approx \rho N / \sqrt{\pi}$. Equation (36) can also be expressed in the coordinate space spanned by $\{a_{\alpha, 1}^\dagger |\text{vac}\rangle, a_{\alpha, 2}^\dagger |\text{vac}\rangle\}$ as

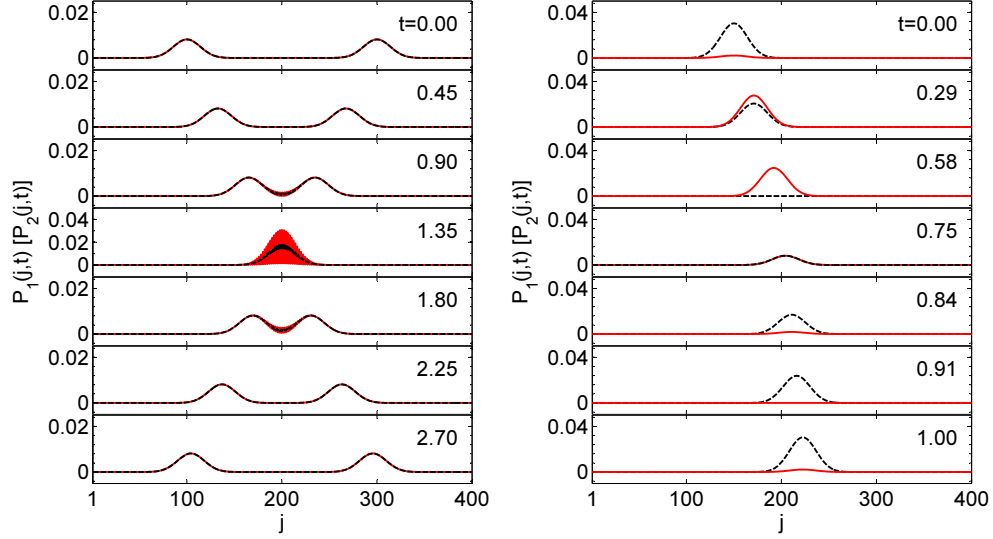


FIG. 2. (Color online) The Dirac probabilities $P_1(j, t)$ (black dashed line) and $P_2(j, t)$ (red solid line) of a particle, initially located in the state $|\Psi(N_A, N_B, \phi_A, \phi_B, 0)\rangle$ for a system with $N = 400$, $\gamma = 0.05$, $J = 0.10$, $\kappa = 1.00$, and $\rho = 0.05$. We obtain $\theta = \pi/6$ and the time t is in units of $T_D \approx 36.276 \kappa^{-1}$. We plot the Eq. (40) for two cases with (a) $\phi_A = -\phi_B = \pi/2$, $N_A = 100$, $N_B = 300$ and (b) $\phi_A = \phi_B = \pi/2$, $N_A = N_B = 150$. The shapes of all the curves are in agreement with the analysis in the text.

$$|\Psi(N_A, N_B, \phi_A, \phi_B, 0)\rangle \approx \sqrt{\frac{\rho}{4\sqrt{\pi} \cos \theta}} \quad (37)$$

$$\times \sum_{\alpha=1}^N \left[e^{-\rho^2(j-N_A)^2/2} e^{i\phi_A j} \left(e^{i\theta/2} a_{\alpha,1}^\dagger - e^{-i\theta/2} a_{\alpha,2}^\dagger \right) \right.$$

$$\left. + e^{-\rho^2(j-N_B)^2/2} e^{i\phi_B j} \left(e^{-i\theta/2} a_{\alpha,1}^\dagger + e^{i\theta/2} a_{\alpha,2}^\dagger \right) \right] |\text{vac}\rangle,$$

which involves both eigenmodes ($\sigma = \pm$) and actually composed of four wavepackets with centers at N_A th and N_B th sites of the legs 1 and 2, and with the velocities ϕ_A and ϕ_B , respectively. To investigate the dynamics of the Dirac norm, substituting

$$f_{k,+} = \frac{1}{\sqrt{\Omega}} e^{-(k-\phi_A)^2/(2\rho^2)} e^{-i(k-\phi_A)N_A}, \quad (38a)$$

$$f_{k,-} = \frac{1}{\sqrt{\Omega}} e^{-(k-\phi_B)^2/(2\rho^2)} e^{-i(k-\phi_B)N_B}, \quad (38b)$$

into Eq. (35), we have

$$|\Psi(N_A, N_B, \phi_A, \phi_B, t)\rangle^2 = \sec \theta + \quad (39)$$

$$e^{-(\phi_A - \phi_B)^2/(4\rho^2)} e^{-\rho^2(N_B - N_A)^2/4} \sin(2\pi t/T_D - \varphi_{AB}) \tan \theta.$$

where $\varphi_{AB} = (N_A + N_B)(\phi_A - \phi_B)/2$. We note that if the two wavepackets of Eq. (36) are well separate in k or α space initially (wavepackets orthogonal in k or α space), the weighted exponential factor becomes zero, then the probability is always conserved in the evolution even they meet each other in the coordinate space

α . This indicates that for states having additional orthogonal modes, Hermitian like behavior still holds even multiple eigenmodes are involved.

To show more detailed propagation behavior, we study the profile of $P_\ell(j, t)$ ($\ell = 1, 2$), where

$$P_\ell(j, t) = |\langle \text{vac} | a_{j,\ell} | \Psi(N_A, N_B, \phi_A, \phi_B, t) \rangle|^2, \quad (40)$$

It is a convenient way to investigate the dynamical properties from two typical cases: (a) $\phi_A = -\phi_B = \pi/2$, $|N_A - N_B| \gg 2\sqrt{\ln 2}/\rho$ and (b) $\phi_A = \phi_B = \pi/2$, $N_A = N_B$. In case (a), the situation corresponds to two counter-propagating wavepackets, with the evolved wave function

$$|\Psi(N_A, N_B, \pi/2, -\pi/2, t)\rangle = \frac{1}{\sqrt{\Omega}} \quad (41)$$

$$\times \sum_k \left[e^{-i\Delta t} e^{-(k-\pi/2)^2/(2\rho^2)} e^{-i(k-\pi/2)(N_A+2\kappa t)} \bar{a}_{k,+} \right.$$

$$\left. + e^{i\Delta t} e^{-(k+\pi/2)^2/(2\rho^2)} e^{-i(k+\pi/2)(N_B-2\kappa t)} \bar{a}_{k,-} \right] |\text{vac}\rangle$$

$$= |\Psi'(N_A + 2\kappa t, N_B - 2\kappa t, \pi/2, -\pi/2, 0)\rangle,$$

where the approximation of Taylor expansions for $\cos k$ around $\pm\pi/2$ are used for two wavepackets and $|\Psi'\rangle$ represents the superposition of two wavepackets as state $|\Psi\rangle$ but with different overall phases. It shows that the evolved state is still the independent nonspreading wavepackets. Similarly, the evolved wave function for case (b) has the form

$$|\Psi(N_A, N_A, \pi/2, \pi/2, t)\rangle = \frac{1}{\sqrt{\Omega}} \sum_k \left[e^{-(k-\pi/2)^2/(2\rho^2)} e^{-i(k-\pi/2)(N_A+2\kappa t)} (\bar{a}_{k,+} e^{-i\Delta t} + \bar{a}_{k,-} e^{i\Delta t}) \right] |\text{vac}\rangle. \quad (42)$$

It has more clear profile in the coordinate space ℓ , i.e.

$$|\Psi(N_A, N_A, \pi/2, \pi/2, t)\rangle \approx \sum_{\ell=1,2} g_\ell(t) \sum_{j=1}^N e^{-\rho^2[j-(N_A+2\kappa t)]^2/2} e^{ij\pi/2} a_{j,\ell}^\dagger |\text{vac}\rangle, \quad (43)$$

where

$$g_\ell(t) = \sqrt{\frac{\rho}{\sqrt{\pi} \cos \theta}} \times \begin{cases} \cos(\pi t/T_D - \theta/2), \ell = 1 \\ i \sin(\pi t/T_D + \theta/2), \ell = 2 \end{cases}. \quad (44)$$

Obviously, it represents two breathing shape-invariant wavepackets propagating along two legs of the ladder with the breathing period T_D . Furthermore, the Dirac norm $P_\ell^s = \sum_j P_\ell(j, t)$ ($\ell = 1, 2$) and $P_T^s = P_1^s + P_2^s$ can be obtained as the form

$$P_1^s = \cos^2(\pi t/T_D - \theta/2) / \cos \theta, \quad (45)$$

$$P_2^s = \sin^2(\pi t/T_D + \theta/2) / \cos \theta, \quad (46)$$

$$P_T^s = \sec \theta + \tan \theta \sin(2\pi t/T_D). \quad (47)$$

As mentioned in the introduction, the profile of the evolved wave function $P_\ell(j, t)$ can be observed in experiment. In practice, the quantum-optical analogy has been employed to visualize the dynamics in the non-Hermitian system [18–20]. In this context, the light intensity corresponds to $P_\ell(j, t)$ (for a review, see [16]) and the profile corresponds to the light intensity distribution along its propagation direction.

It follows that a manifestation of the non-Hermitian nature of H_{Ladd} is represented by the relative phase θ between the breathing oscillations of the two legs, which also leads to the time-dependent Dirac probability. The profiles of the evolved wave functions and the Dirac norms are plotted in Figs. 2 and 3. We can see that in case (a) the evolved wavepackets propagate independently and the Dirac norms are preserving. It indicates that although the Hamiltonian is non-Hermitian, due to the quasi-canonical commutation relations which is a direct consequence of the Hermitian connection structure between clusters, it acts as a Hermitian ladder for some initial state. In contrast, the dynamics of case (b) differs drastically from the Hermitian case and the Dirac norm is no long preserved. Further, the phase difference between the breathing oscillations on the two legs can also be observed in case (b).

IV. SUMMARY AND DISCUSSIONS

In summary, we show in this paper within the context of a class of non-Hermitian lattice systems, which

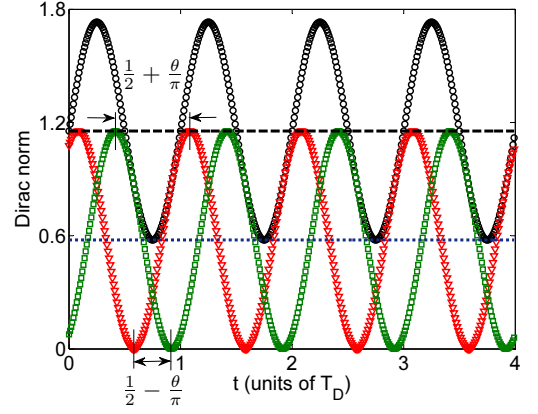


FIG. 3. (Color online) The Dirac norms $P_1^s(t)$, $P_2^s(t)$ (blue dotted line) and $P_T^s(t)$ (black dashed line) for the case of $\phi_A = -\phi_B = \pi/2$ [as in Fig. 2(a)]. The Dirac norms $P_1^s(t)$ (red triangle), $P_2^s(t)$ (green square) and $P_T^s(t)$ (black circle) for the case of $\phi_A = \phi_B = \pi/2$ [as in Fig. 2(b)]. All the parameters are the same as in Fig. 2. The phase difference $\theta = \pi/6$ and also the quasi-canonical commutation relations $\sec \theta \approx 1.155$ are indicated. The shapes of all the curves are in agreement with the analysis in the text.

consist of a set of isomorphic pseudo-Hermitian clusters combined in a Hermitian manner, that Hermitian like dynamics could be observed in such non-Hermitian systems, including the property that the time evolution is Dirac probability preserving. As an application, we investigate a concrete network, a \mathcal{PT} -symmetric ladder, composed of many pseudo-Hermitian dimers. It is shown that it acts as a Hermitian system in the following sense: besides the reality of the spectrum and probability preserving, the propagation of certain wavepackets exhibit the same behavior as that in a Hermitian ladder. Our finding indicates that the reality of the spectrum as well as the Dirac probability preserving dynamics can occur in a system that violating the axiom of Hermiticity. This will pave the way for the development of descriptions of quantum system and provide a topic of considerable interest in a wide range of subjects.

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- [1] C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998).
 - [2] J. Wu and M. Znojil, eds., Proc. Workshop, ‘Pseudo-Hermitian Hamiltonians in Quantum Physics IX’ (Hangzhou, June 2010), Int. J. Theor. Phys. **50**, 953-1333 (2011) and other proceedings in this series.
 - [3] A. Mostafazadeh, J. Math. Phys. **43**, 205 (2002).
 - [4] A. Mostafazadeh and A. Batal, J. Phys. A: Math. Gen. **37**, 11645 (2004).
 - [5] Z. Ahmed, Phys. Lett. A **282**, 343 (2001); Phys. Lett. A **286**, 30 (2001); Phys. Rev. A **64**, 042716 (2001).
 - [6] M. V. Berry, J. Phys. A **31**, 3493 (1998); Czech. J. Phys. **54**, 1039 (2004).
 - [7] W. D. Heiss, Phys. Rep. **242**, 443 (1994); J. Phys. A: Math. Gen. **37**, 2455 (2004);
 - [8] H. F. Jones, J. Phys. A: Math. Gen. **38**, 1741 (2005); Phys. Rev. D **76**, 125003 (2007); Phys. Rev. D **78**, 065032 (2008).
 - [9] J. G. Muga, J. P. Palao, B. Navarro, and I. L. Egusquiza, Phys. Rep. **395**, 357 (2004).
 - [10] C. M. Bender, D. C. Brody, and H. F. Jones, Phys. Rev. Lett. **89**, 270401 (2002).
 - [11] C. M. Bender, Rep. Prog. Phys. **70**, 947 (2007).
 - [12] P. Dorey, C. Dunning, and R. Tateo, J. Phys. A: Math. Theor. **40**, R205 (2007).
 - [13] A. Mostafazadeh, Int. J. Geom. Meth. Mod. Phys. **7**, 1191 (2010).
 - [14] A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou, and D. N. Christodoulides, Phys. Rev. Lett. **103**, 093902 (2009).
 - [15] C. E. Rüter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev, and D. Kip, Nature Physics **6**, 192 (2010).
 - [16] S. Longhi, Laser & Photon. Rev. **3**, 243 (2009).
 - [17] S. Klaiman, U. Günther, and N. Moiseyev, Phys. Rev. Lett. **101**, 080402 (2008).
 - [18] K. G. Makris, R. El-Ganainy, D. N. Christodoulides, and Z. H. Musslimani, Phys. Rev. Lett. **100**, 103904 (2008); Phys. Rev. A **81**, 063807 (2010).
 - [19] Z. H. Musslimani, K. G. Makris, R. El-Ganainy, and D. N. Christodoulides, Phys. Rev. Lett. **100**, 030402 (2008).
 - [20] R. El-Ganainy, K. G. Makris, D. N. Christodoulides, and Z. H. Musslimani, Opt. Lett. **32**, 2632 (2007).
 - [21] S. Longhi, Phys. Rev. Lett. **103**, 123601 (2009).
 - [22] M. Znojil, J. Phys. A: Math. Theor. **40**, 13131 (2007); J. Phys. A: Math. Theor. **41**, 292002 (2008); Phys. Rev. A **82**, 052113 (2010); J. Phys. A: Math. Theor. **44**, 075302 (2011).
 - [23] O. Bendix, R. Fleischmann, T. Kottos, and B. Shapiro, Phys. Rev. Lett. **103**, 030402 (2009).
 - [24] S. Longhi, Phys. Rev. B **80**, 235102 (2009); Phys. Rev. B **81**, 195118 (2010); Phys. Rev. A **82**, 032111 (2010); Phys. Rev. B **82**, 041106(R) (2010).
 - [25] Y. N. Joglekar, D. Scott, M. Babbey, and A. Saxena Phys. Rev. A **82**, 030103(R) (2010).
 - [26] Y. N. Joglekar and A. Saxena, Phys. Rev. A **83**, 050101(R) (2011); D. D. Scott and Y. N. Joglekar Phys. Rev. A **83**, 050102(R) (2011).
 - [27] T. Deguchi and P. K. Ghosh J. Phys. A: Math. Theor. **42**, 475208 (2009).
 - [28] G. L. Giorgi, Phys. Rev. B **82**, 052404 (2010).
 - [29] C. Korff and R. Weston, J. Phys. A: Math. Theor. **40**, 8845 (2007).
 - [30] O. A. Castro-Alvaredo and A. Fring, J. Phys. A: Math. Theor. **42**, 465211 (2009).
 - [31] H. Zhong, W. Hai, G. Lu, and Z. Li, Phys. Rev. A **84**, 013410 (2011).
 - [32] L. B. Drissi, E. H. Saidi, and M. Bousmina, J. Math. Phys. **52**, 022306 (2011).
 - [33] Özlem Yeşiltaş, J. Phys. A: Math. Theor. **44**, 305305 (2011).
 - [34] F. Bagarello, M. Znojil, J. Phys. A: Math. Theor. **44**, 415305 (2011).
 - [35] L. Jin and Z. Song, Commun. Theor. Phys. **54**, 73 (2010).
 - [36] L. Jin and Z. Song, Phys. Rev. A **80**, 052107 (2009).
 - [37] R. G. Littlejohn, Phys. Rep. **138**, 193 (1986).
 - [38] T. J. Osborne and N. Linden, Phys. Rev. A **69**, 052315 (2004).
 - [39] T. Shi, Y. Li, Z. Song, and C. P. Sun, Phys. Rev. A **71**, 032309 (2005).
 - [40] S. Yang, Z. Song, and C. P. Sun, Phys. Rev. B **73**, 195122 (2006).
 - [41] L. Jin and Z. Song, Phys. Rev. A **79**, 042341 (2009).
 - [42] X. Z. Zhang, L. Jin, and Z. Song e-print arXiv:1106.0087 (2011).
 - [43] W. Kim, L. Covaci, and F. Marsiglio, Phys. Rev. B **74**, 205120 (2006).